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Sylvie Benzoni-Gavage[⋆], Jean-François Coulombel[†] & Nikolay Tzvetkov[‡]

⋆ Université de Lyon, Université Lyon 1, INSA de Lyon, Ecole Centrale de Lyon
CNRS, UMR5208, Institut Camille Jordan, 43 blvd du 11 novembre 1918
F-69622 Villeurbanne-Cedex, France

† CNRS, Université Lille 1, Laboratoire Paul Painlevé (UMR CNRS 8524)
and EPI SIMPAF of INRIA Lille Nord Europe
Cité scientifique, Bâtiment M2, 59655 Villeneuve d'Ascq Cedex, France

‡ Département de Mathématiques (UMR CNRS 8088), Université de Cergy-Pontoise
2 avenue Adolphe Chauvin, 95302 Cergy-Pontoise Cedex, France

Emails: `benzoni@math.univ-lyon1.fr`, `jfcoulom@math.univ-lille1.fr`
`nikolay.tzvetkov@u-cergy.fr`

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Abstract

Nonlocal generalizations of Burgers equation were derived in earlier work by Hunter [Contemp. Math. 1989], and more recently by Benzoni-Gavage and Rosini [Comput. Math. Appl. 2009], as weakly nonlinear amplitude equations for hyperbolic boundary value problems admitting linear surface waves. The local-in-time well-posedness of such equations in Sobolev spaces was proved by Benzoni-Gavage [Diff. Int. Eq. 2009] under an appropriate stability condition originally pointed out by Hunter. In this article, it is shown that the latter condition is not only sufficient for well-posedness in Sobolev spaces but also necessary. The main point of the analysis is to show that when the stability condition is violated, nonlocal Burgers equations reduce to second order elliptic PDEs. The resulting ill-posedness result encompasses various cases previously studied in the literature.

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1 Introduction

We consider the Cauchy problem for a nonlocal generalization of the inviscid Burgers equation

$$\partial_t u + \partial_x \mathscr{Q}(u) = 0\,, \quad u_{|t=0} = u_0\,, \tag{1}$$

where $x \in \mathbb{R}$ is the space variable, t denotes the time variable, and $\mathscr{Q}$ is a quadratic operator acting nonlocally in the Fourier variables. For any real-valued function u, say in the Schwartz class $\mathscr{S}(\mathbb{R})$, the function $\mathscr{Q}(u)$ is defined by the integral formula

$$\widehat{\mathscr{Q}(u)}(k) := \int_{\mathbb{R}} \Lambda(k-\ell,\ell)\,\widehat{u}(k-\ell)\,\widehat{u}(\ell)\,\mathrm{d}\ell\,, \tag{2}$$

where the kernel $\Lambda \in L^\infty(\mathbb{R}\times\mathbb{R};\mathbb{C})$ is such that $\Lambda(-k,-\ell) = \overline{\Lambda(k,\ell)}$ for all $(k,\ell) \in \mathbb{R}^2$, which ensures that $\mathscr{Q}(u)$ is also real-valued. Actually, the formula (2) makes sense for a much larger class of functions than those in $\mathscr{S}(\mathbb{R})$, and in particular for functions in the Sobolev space $H^1(\mathbb{R})$, see [2] and below for more details.

The usual (inviscid) Burgers equation would correspond to a constant kernel Λ. Apart from this 'degenerate' case, equations as in (1)-(2) with genuine nonlocal effects arise in particular as amplitude equations for weakly nonlinear waves [8, 1, 3]. The specific form of the kernel Λ of course heavily depends on the underlying physical framework. However, two very general properties are

(i) **symmetry:** $\Lambda(k,\ell) = \Lambda(\ell,k)\,,\ \forall\, k,\ \ell \in \mathbb{R}$,

(ii) **reality:** $\Lambda(-k,-\ell) = \overline{\Lambda(k,\ell)}\,,\ \forall\, k,\ \ell \in \mathbb{R}$.

The former can always be obtained by redefining Λ properly, and, as already mentioned, the latter is important for $\mathscr{Q}$ to transform real-valued functions into real-valued functions. We shall make two more peculiar assumptions - which are satisfied in the examples quoted below -, namely

(iii) **homogeneity:** $\Lambda(\alpha\, k, \alpha\, \ell) = \Lambda(k,\ell)\,,\ \forall\, k,\ \ell \in \mathbb{R}\,,\ \forall\, \alpha > 0$.

(iv) **regularity:** $\Lambda \in \mathscr{C}^1(\{(k,\ell)\,;\ k\ell(k+\ell) \neq 0\})$ and admits $\mathscr{C}^1$ extensions to the closed sectors $\mathbb{R}^+ \times \mathbb{R}^+$ (and its symmetric counterpart $\mathbb{R}^- \times \mathbb{R}^-$), and $\{(k,\ell) \in \mathbb{R}^+ \times \mathbb{R}^-\,;\ k+\ell \geq 0\}$, $\{(k,\ell) \in \mathbb{R}^+ \times \mathbb{R}^-\,;\ k+\ell \leq 0\}$ (and their symmetric counterparts).

It has been shown in [2] (also see [9, 13]) that for kernels having the properties (i)-(ii)-(iii)-(iv), a sufficient condition for the well-posedness of (1) in Sobolev spaces (of high enough index) is the following one

(v) **stability:** $\Lambda(1,0^+) = \Lambda(-1,0^+)$.

This condition was already pointed out by Hunter in [8]. Our purpose here is to show that, as conjectured by Hunter, the stability condition (v) is also necessary for well-posedness in Sobolev spaces.

A typical kernel obviously satisfying the properties (i)-(ii)-(iii)-(iv) but violating (v) is

$$\Lambda_0(k,\ell) := \mathrm{sgn}(k)\,\mathrm{sgn}(\ell)\,, \tag{3}$$

where by $\mathrm{sgn}(k)$ we mean 1 if $k > 0$ and -1 if $k < 0$ (we do not need a definition for $k = 0$). The associated quadratic functional $\mathscr{Q}_0$ is given by

$$\mathscr{Q}_0(u) = -2\,\pi\,\mathscr{H}(u)^2\,,$$

where $\mathscr{H}$ denotes the Hilbert transform, defined in Fourier variables by

$$\widehat{\mathscr{H}(u)}(\xi) \;=\; -\,i\,\mathrm{sgn}(\xi)\,\widehat{u}(\xi)\,.$$

A very close alternative example

$$\Lambda(k,\ell) = 1 + \mathrm{sgn}(k)\,\mathrm{sgn}(\ell) \tag{4}$$

was considered in [8, p. 199], and a seemingly different example is

$$\Lambda(k,\ell) = -\frac{i}{2}\,(\mathrm{sgn}(k) + \mathrm{sgn}(\ell))\,, \tag{5}$$

corresponding to the equation studied in [4]. It turns out that all these examples can somehow be reduced to a complex Burgers equation. This assertion will be justified in Section 5. A much more complicated kernel was obtained in [3], which satisfies (i)-(ii)-(iii)-(iv) and apparently not (v). It is therefore of interest to consider well-posedness issues in general. The 'simple' kernel Λ_0 in (3) will serve as a model for our study (see Section 4), and we will eventually obtain an ill-posedness result for general kernels under the conditions (i)-(ii)-(iii)-(iv) and

(nv) **instability:** $\Lambda(1,0^+) \neq \Lambda(-1,0^+)$.

Our main result is indeed the following.

Theorem 1. *Under the conditions* (i)-(ii)-(iii)-(iv)-(nv) *on* Λ, *assuming moreover that*

(iv') Λ *is continuous across the line* $k+\ell=0$,

the Cauchy problem (1) *for* $\mathscr{Q}$ *defined in* (2) *is ill-posed in* $H^m(\mathbb{R})$, $m \geq 4$. *More precisely, there exists a dense subset* $\mathscr{O} \subset H^m(\mathbb{R})$ *such that for any initial data* $u_0 \in \mathscr{O}$, *for any* $T>0$, *the Cauchy problem* (1) *has no solution* $u \in \mathscr{C}([-T,T];H^m(\mathbb{R}))$.

Remark 1. *The additional assumption* (iv') *is obviously satisfied by the examples in* (3), (4), *and* (5). *It also turns out to be true for the kernel associated with 'surface acoustic waves' in elasticity, as we can see on its explicit form given in [8, p. 201]* ($\Lambda(k,-k)=0$), *but of course our present theorem does not apply in this case since the stability condition* (v) *is satisfied (hence well-posedness by the main result in [2]). General kernels as in [8, 3] have a (purely imaginary) jump across the line* $k+\ell=0$, *and reasons why this jump could be zero need further investigation. Anyway, the failure of* (iv') *is more likely to worsen ill-posedness than to compete with it.*

The ill-posedness result in Theorem 1 is of course a serious obstacle for the justification on weakly nonlinear geometric optics expansions when the resulting amplitude equation does not satisfy the stability condition (v).

The paper is organized as follows. In Section 2 we prove, under the only assumptions (i)-(ii) on $\Lambda \in L^\infty(\mathbb{R}\times\mathbb{R};\mathbb{C})$, a Cauchy–Kovalevskaya type result for equation (1). This first result shows that nonexistence of a local in time solution in Sobolev spaces for (1) can be achieved for at most a dense subset of initial data. In Section 3 we recall from [2] the well-posedness result known under (i)-(ii)-(iii)-(iv)-(v), and we provide evidence that the energy method fails when (v) is violated. It turns out, however, that the blow-up of inner products is not strong enough to contradict well-posedness by the method used in the theory of dispersive equations, see e.g. [5] (and also [19] for a comprehensive overview). So we proceed differently, and show that when one has (nv), Eq. (1) amounts to a second order elliptic PDE. This is basically what was done in [8], [4] on the specific examples (4) and (5) respectively. Here we obtain an elliptic principal part, together with lower order pseudo-differential remainder terms, for general kernels. Section 4 is devoted, mainly for clarity, to the special case of the kernel Λ_0 mentioned above. The general case is dealt with in Section 5.

All along the paper, we use the following notations. The Fourier transform $\mathscr{F}u = \widehat{u}$ of a function u is defined using the convention that

$$\widehat{u}(\xi) \;=\; \int \mathrm{e}^{-i\,x\,\xi}\, u(x)\,\mathrm{d}x$$

whenever this formula is meaningful, so that the inverse formula reads

$$u(x) \;=\; \frac{1}{2\,\pi} \int \mathrm{e}^{i\,x\,\xi}\, \widehat{u}(\xi)\,\mathrm{d}\xi\,.$$

The 'japanese bracket' is used for $\langle k\rangle = (1+k^2)^{1/2}$, and for $s \geq 0$,

$$H^s(\mathbb{R}) \;=\; \{u \in L^2(\mathbb{R})\,;\; \langle\cdot\rangle^s\,\widehat{u} \in L^2\}$$

is equipped with the usual norm defined by $\|u\|_{H^s} \;=\; \|\langle\cdot\rangle^s\,\widehat{u}\|_{L^2}/\sqrt{2\,\pi}$.
The brackets $[\;;\;]$ will stand for commutators (for two operators A and B, $[A;B] = AB - BA$ as long as this is meaningful). The symbol $\lesssim$ means $\leq$ up to a harmless, multiplicative constant.

2 Well-posedness in the analytic framework

Let us define the following scale of (real) vector spaces, for $\rho > 0$,

$$E_\rho := \left\{u \in L^2(\mathbb{R};\mathbb{R})\,;\; \langle\cdot\rangle \mathrm{e}^{\rho|\cdot|}\,\widehat{u} \in L^2\right\}$$

equipped with the natural norm

$$\|u\|_{E_\rho} := \left(\int_{\mathbb{R}} \langle k\rangle^2\,\mathrm{e}^{2\,\rho\,|k|}\,|\widehat{u}(k)|^2\,\mathrm{d}k\right)^{1/2}\,.$$

These are Hilbert spaces, and for $\rho' \leq \rho$, the space E_ρ is imbedded in $E_{\rho'}$ thanks to the straightforward inequality $\|u\|_{E_{\rho'}} \leq \|u\|_{E_\rho}$. By the Fourier inverse formula and Cauchy-Schwarz inequality we see that functions pertaining to E_ρ are analytic and admit a holomorphic extension to a horizontal strip containing the real axis in the complex plane. Conversely, by Cauchy's theorem we find that any analytic function on $\mathbb{R}$ belongs to some E_ρ for $\rho > 0$ small enough. In such an analytic framework, the following well-posedness result is rather standard for first-order equations.

Proposition 1 (Local well-posedness for analytic data)**.** *Let $\Lambda \in L^\infty(\mathbb{R}\times\mathbb{R};\mathbb{C})$ satisfy* (i)-(ii) *and $\mathscr{Q}$ be defined by* (2)*. Then for all $\rho_0 > 0$, and for all $u_0 \in E_{\rho_0}$, there exist a constant $\kappa > 0$ and a unique function u belonging to $\mathscr{C}^1(]-\kappa\,(\rho_0-\rho),\kappa\,(\rho_0-\rho)[;E_\rho)$ for every positive $\rho < \rho_0$, which solves* (1) *on the time interval* $]-\kappa\,\rho_0,\kappa\,\rho_0[$.

The proof of Proposition 1 relies on a continuity estimate for the quadratic operator $\mathscr{Q}$ in the spaces E_ρ, and more precisely on the following elementary result.

Lemma 1. *Let $\Lambda \in L^\infty(\mathbb{R}\times\mathbb{R};\mathbb{C})$ satisfy* (i), (ii)*. The formula*

$$\widehat{\mathscr{B}(u,v)}(k) := \int_{\mathbb{R}} \Lambda(k-\ell,\ell)\,\widehat{u}(k-\ell)\,\widehat{v}(\ell)\,\mathrm{d}\ell \tag{6}$$

defines a symmetric bilinear operator on $E_\rho\times E_\rho$ for all $\rho>0$, and there exists a numerical constant $C_0>0$, independent of ρ, such that there holds

$$\forall\, u,v\in E_\rho\,,\quad \|\mathscr{B}(u,v)\|_{E_\rho} \le C_0\,\|\Lambda\|_{L^\infty(\mathbb{R}^2)}\,\|u\|_{E_\rho}\,\|v\|_{E_\rho}\,. \tag{7}$$

Proof. It is straightforward to check that for $u,v\in E_\rho$, the function $\widehat{\mathscr{B}(u,v)}$ defined by (6) is measurable and square integrable. By inverse Fourier transform this defines $\mathscr{B}(u,v)\in L^2(\mathbb{R})$ in a unique way. Furthermore, the relation

$$\forall\, k\in\mathbb{R}\,,\quad \widehat{\mathscr{B}(u,v)}(-k) = \overline{\widehat{\mathscr{B}(u,v)}(k)}$$

is obtained by a simple change of variables from (ii) and the fact that both u and v are real-valued, hence $\mathscr{B}(u,v)$ is real-valued.

Let us now estimate the quantity

$$I := \int_{\mathbb{R}} \langle k\rangle^2\,\mathrm{e}^{2\,\rho\,|k|}\,\big|\widehat{\mathscr{B}(u,v)}(k)\big|^2\,\mathrm{d}k\,.$$

By the triangle inequality, we first obtain

$$I \le \|\Lambda\|_{L^\infty}^2 \int_{\mathbb{R}} \left\{\int_{\mathbb{R}} \langle k\rangle\,\mathrm{e}^{\rho\,|k-\ell|}\,|\widehat{u}(k-\ell)|\,\mathrm{e}^{\rho\,|\ell|}\,|\widehat{v}(\ell)|\,\mathrm{d}\ell\right\}^2\,\mathrm{d}k\,.$$

Now we use the inequality

$$\langle k\rangle \le \sqrt{2}\,\left\{\langle \ell\rangle + \langle k-\ell\rangle\right\}$$

to derive

$$\begin{aligned} I \le{}& 4\,\|\Lambda\|_{L^\infty}^2 \int_{\mathbb{R}} \left\{\int_{\mathbb{R}} \langle k-\ell\rangle\,\mathrm{e}^{\rho\,|k-\ell|}\,|\widehat{u}(k-\ell)|\,\mathrm{e}^{\rho\,|\ell|}\,|\widehat{v}(\ell)|\,\mathrm{d}\ell\right\}^2\,\mathrm{d}k \\ &+ 4\,\|\Lambda\|_{L^\infty}^2 \int_{\mathbb{R}} \left\{\int_{\mathbb{R}} \mathrm{e}^{\rho\,|k-\ell|}\,|\widehat{u}(k-\ell)|\,\langle \ell\rangle\,\mathrm{e}^{\rho\,|\ell|}\,|\widehat{v}(\ell)|\,\mathrm{d}\ell\right\}^2\,\mathrm{d}k\,. \end{aligned}$$

It remains to use the classical convolution estimate $L^1 * L^2 \to L^2$, and we obtain

$$I \le 4\,\|\Lambda\|_{L^\infty}^2 \left(\|u\|_{E_\rho}^2\,\|\mathrm{e}^{\rho\,|\cdot|}\,\widehat{v}\|_{L^1(\mathbb{R})}^2 + \|\mathrm{e}^{\rho\,|\cdot|}\,\widehat{u}\|_{L^1(\mathbb{R})}^2\,\|v\|_{E_\rho}^2\right)\,.$$

Noting that

$$\|\mathrm{e}^{\rho\,|\cdot|}\,\widehat{v}\|_{L^1(\mathbb{R})} \le \sqrt{\pi}\,\|u\|_{E_\rho}$$

by the Cauchy–Schwarz inequality, we get the estimate in (7) for $\|\mathscr{B}(u,v)\|_{E_\rho} = \sqrt{I}$. $\square$

Proof of Proposition 1. It will follow from the abstract Cauchy–Kovalevskaya theorem, for which we refer e.g. to [15, 16, 17]. More precisely, (1) can be recast as

$$\frac{\mathrm{d}u}{\mathrm{d}t} = F(u(t)) \,, \quad u(0) = u_0 \,, \quad F(u) := -2\,\mathscr{B}(u, \partial_x u) \,.$$

Let us observe that for $\rho > 0$ and $u \in E_\rho$, the derivative $u' = \partial_x u$ belongs to any $E_{\rho'}$, $\rho' < \rho$, with the estimate

$$\|u'\|_{E_{\rho'}} \le \frac{\mathrm{e}^{-1}}{\rho - \rho'} \, \|u\|_{E_\rho} \,.$$

Therefore, as a consequence of Lemma 1, there exists a constant C_0 such that for all $0 < \rho' < \rho$, for all $u, v \in E_\rho$,

$$\|F(u) - F(v)\|_{E_{\rho'}} \le \frac{C_0 \left(\|u\|_{E_\rho} + \|v\|_{E_\rho} \right)}{\rho - \rho'} \, \|u - v\|_{E_\rho} \,.$$

In particular, for all $0 < \rho' < \rho$, F is continuous (and locally Lipschitz) from E_ρ to $E_{\rho'}$. These are all the conditions required to apply the abstract Cauchy–Kovalevskaya theorem in the scale of spaces E_ρ. We refer the reader to the above mentioned references for more details. □

Proposition 1 shows that the local-in-time well-posedness of (1) in the analytic framework is basically independent of Λ. A natural question is now to understand the well-posedness of (1) in the framework of Sobolev spaces. This is in some sense a stability problem. Given an initial condition $u_0 \in H^m(\mathbb{R})$, $m \ge 2$, and a sequence (u_0^n) in, say, E_1 that converges towards u_0 in $H^m(\mathbb{R})$, does there exist a positive time $T > 0$ such that the sequence of solutions (u^n) to (1) is bounded in $\mathscr{C}([-T,T]; H^m(\mathbb{R}))$? If the answer is positive, then we should be able to construct a local-in-time solution to (1) by an approximation and compactness argument. It turns out that such a stability property in $H^m(\mathbb{R})$ heavily depends on the kernel Λ, as made precise in the following paragraphs. Let us simply note that the functional setting $H^m(\mathbb{R})$, $m \ge 2$, is quite natural for studying (1), because it is the one where hyperbolic equations are known to be locally well-posed in one space dimension.

3 Well-posedness in Sobolev spaces: a reminder

Let us first recall the following well-posedness result from [2].

Theorem 2. *Let Λ satisfy conditions* (i), (ii), (iii), (iv), (v). *Then for all $R > 0$ there exists $T > 0$ such that for all $u_0 \in H^2(\mathbb{R})$ with $\|u_0\|_{H^2(\mathbb{R})} \le R$, there exists a unique $u \in \mathscr{C}([-T,T]; H^2(\mathbb{R}))$ solution to* (1) *with $u_{|t=0} = u_0$. Furthermore, the mapping $u_0 \in H^2(\mathbb{R}) \mapsto u \in \mathscr{C}([-T,T]; H^2(\mathbb{R}))$ is continuous on every ball of $H^2(\mathbb{R})$. If $u_0 \in H^m(\mathbb{R})$, $m \ge 2$, then the solution u belongs to $\mathscr{C}([-T,T]; H^m(\mathbb{R}))$.*

The method of proof crucially relies on the energy method, and more specifically on the *a priori* estimates

$$\left| \langle \partial_x^m u, \partial_x^{m+1} \mathscr{Q}(u) \rangle_{L^2} \right| \le C(\Lambda) \, \|\mathscr{F}(\partial_x u)\|_{L^1} \, \|u\|_{H^m}^2 \,, \tag{8}$$

valid for $m = 0, 1, 2, 3$ under the conditions (i), (ii), (iii), (iv), (v). Now the next result shows that the failure of (v) entails the failure of the energy method, namely it makes possible the 'blow-up' of the inner product $\langle \partial_x^m u, \partial_x^{m+1} \mathscr{Q}(u) \rangle_{L^2}$ for any $m \geq 2$.

Proposition 2. *Let Λ satisfy conditions* (i), (ii), (iii), (iv), *and* (nv). *Then for all integer $m \geq 2$, there exists a sequence of real-valued functions $(u_n)_{n \in \mathbb{N}}$ in the Schwartz class such that*

$$\forall\, n \in \mathbb{N}\,, \quad \|u_n\|_{H^m} \leq \frac{1}{c}\,, \quad \langle \partial_x^m u_n, \partial_x^{m+1} \mathscr{Q}(u_n) \rangle_{L^2} \leq -\, c\, n$$

for a positive constant c independent of n.

Proof. We first prove the result for $\Lambda = \Lambda_0$, the model kernel defined by

$$\Lambda_0(k, \ell) = \operatorname{sgn}(k) \operatorname{sgn}(\ell)\,,$$

and the associated quadratic operator $\mathscr{Q} = \mathscr{Q}_0$, a case that will serve as a building block for the general one. By definition, for all $u \in \mathscr{S}(\mathbb{R})$, and all integer m,

$$\langle \partial_x^m u_n, \partial_x^{m+1} \mathscr{Q}(u_n) \rangle_{L^2} \;=\; \frac{1}{2\,\pi} \iint i\, \xi^{2\,m+1}\, \widehat{u}(-\xi) \operatorname{sgn}(\xi - \ell) \operatorname{sgn}(\ell)\, \widehat{u}(\xi - \ell)\, \widehat{u}(\ell)\, \mathrm{d}\ell\, \mathrm{d}\xi\,.$$

A way to make this inner product large (in absolute value) is to choose a u with basically two frequencies, a low one and a large one. To be precise, let us define u_n by

$$\widehat{u_n}(\xi) \;=\; \begin{cases} i \operatorname{sgn}(\xi)\,, & \text{if } |\xi| \in [0, 1/n]\,, \\ i \operatorname{sgn}(\xi)\, n^{-2s+1/2}\,, & \text{if } |\xi| \in [n^2, n^2 + 1/n]\,, \\ 0 & \text{otherwise.} \end{cases}$$

This u_n, obviously not in $\mathscr{S}(\mathbb{R})$, should actually be modified by using smooth cut-off functions. Then there would be additional, harmless terms in what follows, which we omit for simplicity. Then

$$\|u_n\|^2_{H^m(\mathbb{R})} \;\leq\; \frac{2}{n}\, \left(\langle n^{-1} \rangle^{2\,m} \;+\; n^{-4\,m+1}\, \langle n^2 + n^{-1} \rangle^{2\,m} \rangle \right)\,,$$

which is uniformly bounded with respect to n, and

$$\begin{aligned} -2\pi\, \langle \partial_x^m u_n, \partial_x^{m+1} \mathscr{Q}(u_n) \rangle_{L^2} \;&\geq\; \int_{n^2}^{n^2+1/n} \int_{n^2}^{n^2+1/n} |\xi|^{2\,m+1}\, n^{-4\,m+1}\, \mathrm{d}\ell\, \mathrm{d}\xi \\ &\geq\; \frac{1}{n^2}\, n^{2\,(2\,m+1)}\, n^{-4\,m+1} \;=\; n\,. \end{aligned}$$

We now turn to a general kernel, and look for $u \in \mathscr{S}(\mathbb{R})$ such that

$$I := -\, 2\pi\, \langle \partial_x^m u, \partial_x^{m+1} \mathscr{Q}(u) \rangle_{L^2} \;=\; -\, \iint i\, \xi^{2m+1}\, \widehat{u}(-\xi)\, \Lambda(\xi - \ell, \ell)\, \widehat{u}(\xi - \ell)\, \widehat{u}(\ell)\, \mathrm{d}\ell\, \mathrm{d}\xi$$

is arbitrarily large, where m is an integer, $m \geq 2$. For obvious symmetry reasons, we may rewrite the integral I as

$$\int_S i\, h_3^{2m+1}\, \Lambda(h_1, h_2)\, \widehat{u}(h_1)\, \widehat{u}(h_2)\, \widehat{u}(h_3)\, \mathrm{d}\sigma(h)$$

with $S = \{h = (h_1, h_2, h_3) \in \mathbb{R}^3\,;\ h_1 + h_2 + h_3 = 0\}$ equipped with the Lebesgue measure σ, or using the definition of S and the symmetry property (i) of Λ,

$$I \;=\; -2 \int_S i\, h_1\, h_3^{2m}\, \Lambda(h_1, h_2)\, \widehat{u}(h_1)\, \widehat{u}(h_2)\, \widehat{u}(h_3)\, \mathrm{d}\sigma(h)$$

$$= - \int_S i\,(\, h_1\, h_3^{2m}\, \Lambda(h_1, h_2) \;+\; h_3\, h_1^{2m}\, \Lambda(h_3, h_2)\,)\, \widehat{u}(h_1)\, \widehat{u}(h_2)\, \widehat{u}(h_3)\, \mathrm{d}\sigma(h) \;=\; J_1 \;+\; J_2 \;+\; J_3$$

with

$$J_1 \;=\; - \int_S i\, \Lambda(h_3, h_2)\,(\, h_1\, h_3^{2m} \;+\; h_3\, h_1^{2m}\,)\, \widehat{u}(h_1)\, \widehat{u}(h_2)\, \widehat{u}(h_3)\, \mathrm{d}\sigma(h)\,,$$

$$J_2 \;=\; - \int_{S;\, |h_3| \le |h_2|} i\, h_1\, h_3^{2m}\,(\, \Lambda(h_1, h_2) \;-\; \Lambda(h_3, h_2)\,)\, \widehat{u}(h_1)\, \widehat{u}(h_2)\, \widehat{u}(h_3)\, \mathrm{d}\sigma(h)\,,$$

$$J_3 \;=\; - \int_{S;\, |h_3| > |h_2|} i\, h_1\, h_3^{2m}\,(\, \Lambda(h_1, h_2) \;-\; \Lambda(h_3, h_2)\,)\, \widehat{u}(h_1)\, \widehat{u}(h_2)\, \widehat{u}(h_3)\, \mathrm{d}\sigma(h)\,.$$

The first one turns out to be bounded if $\|u\|_{H^m}$ is so. Indeed, by Lemma 4 given in the appendix, for $(h_1, h_2, h_3) \in S$,

$$\begin{aligned} |h_1\, h_3^{2m} \;+\; h_3\, h_1^{2m}| \;\le\; & \;|h_1\, h_2^m\, h_3^m| \;+\; |h_1^m\, h_2^m\, h_3| \;+\; |h_1^m\, h_2\, h_3^m| \\ & + \textstyle\sum_{k=2}^{m-1} \binom{m}{k}\, |h_2^k|\,(\, |h_1^{m+1-k}\, h_3^m| \;+\; |h_1^m\, h_3^{m+1-k}|\,)\,, \end{aligned}$$

hence, by Fubini and Cauchy–Schwarz,

$$\begin{aligned} |J_1| \;\le\; & \|\Lambda\|_{L^\infty}\,(\, 3\, \|\mathscr{F} \partial_x u\|_{L^1}\, \|u\|_{H^m}^2 \;+\; 4 \sum_{2 \le k \le (m+1)/2} \binom{m}{k}\, \|\mathscr{F} \partial_x^k u\|_{L^1}\, \|u\|_{H^{m+1-k}}\, \|u\|_{H^m}\,) \\ \le\; & C_m(\|\Lambda\|_{L^\infty})\, \|u\|_{H^m}^3 \end{aligned}$$

since $\|\mathscr{F} \partial_x u\|_{L^1} \;\lesssim\; \|u\|_{H^s}$ for $s > 3/2$. (Note that if m is even, the sum extends to $k \le m/2$ only, and $m > m/2 + 1/2$, while if m is odd, the sum extends to $k \le (m+1)/2$, and $m > (m+1)/2 + 1/2$ because $m \ne 2$ and $m \ge 2$ by assumption.) Clearly J_2 remains bounded too because

$$\begin{aligned} |J_2| \;\le\; & 2\, \|\Lambda\|_{L^\infty} \int_S |h_1|\, |h_2|^m\, |h_3|^m\, |\widehat{u}(h_1)\, \widehat{u}(h_2)\, \widehat{u}(h_3)|\, \mathrm{d}\sigma(h) \\ \le\; & 2\, \|\Lambda\|_{L^\infty}\, \|\mathscr{F} \partial_x u\|_{L^1}\, \|u\|_{H^m}^2 \end{aligned}$$

by Fubini and Cauchy–Schwarz again. We now deal with J_3. By the homogeneity of Λ, we can rewrite it as

$$\begin{aligned} J_3 \;=\; - \int_{S;\, |h_3| > |h_2|} & i\, h_1\, h_3^{2m} \left(\Lambda\Big(-\mathrm{sgn}(h_3) - \frac{h_2}{|h_3|}, \frac{h_2}{|h_3|}\Big) \;-\; \Lambda\Big(\mathrm{sgn}(h_3), \frac{h_2}{|h_3|}\Big) \right) \\ & \times \widehat{u}(h_1)\, \widehat{u}(h_2)\, \widehat{u}(h_3)\, \mathrm{d}\sigma(h)\,. \end{aligned}$$

Depending on the sign $\pm$ of h_2, we may write

$$\begin{aligned}\Lambda\Big(-\mathrm{sgn}(h_3)-\frac{h_2}{|h_3|},\frac{h_2}{|h_3|}\Big)-\Lambda\Big(\mathrm{sgn}(h_3),\frac{h_2}{|h_3|}\Big) \;=\;& \Lambda\Big(-\mathrm{sgn}(h_3)-\frac{h_2}{|h_3|},\frac{h_2}{|h_3|}\Big)\;-\;\Lambda\Big(-\mathrm{sgn}(h_3),0\pm\Big)\\ &+\Lambda\Big(-\mathrm{sgn}(h_3),0\pm\Big)\;-\;\Lambda\Big(\mathrm{sgn}(h_3),0\pm\Big)\\ &+\Lambda\Big(\mathrm{sgn}(h_3),0\pm\Big)\;-\;\Lambda\Big(\mathrm{sgn}(h_3),\frac{h_2}{|h_3|}\Big).\end{aligned}$$

Therefore, we obtain that $J_3 = K_1 + K_2$ where, using Lipchitz bounds for Λ as in [2],

$$|K_1| \,\le\, C(\Lambda)\int_{S;\,|h_3|>|h_2|}|h_1\,h_2\,h_3^{2m-1}\widehat{u}(h_1)\,\widehat{u}(h_2)\,\widehat{u}(h_3)|\,\mathrm{d}\sigma(h)\,,$$

which can be bounded (more easily than J_1) by a constant times $\|u\|_{H^m}^3$, and

$$\begin{aligned}K_2 \;=\;& -\int_{S;\,h_3>h_2>0} i\,h_1\,|h_3|^{2m}\,\big(\Lambda(-1,0+)\,-\,\Lambda(1,0+)\big)\,\widehat{u}(h_1)\,\widehat{u}(h_2)\,\widehat{u}(h_3)\,\mathrm{d}\sigma(h)\\ &-\int_{S;\,h_3>-h_2>0} i\,h_1\,|h_3|^{2m}\,\big(\Lambda(-1,0-)\,-\,\Lambda(1,0-)\big)\,\widehat{u}(h_1)\,\widehat{u}(h_2)\,\widehat{u}(h_3)\,\mathrm{d}\sigma(h)\\ &-\int_{S;\,-h_3>-h_2>0} i\,h_1\,|h_3|^{2m}\,\big(\Lambda(1,0-)\,-\,\Lambda(-1,0-)\big)\,\widehat{u}(h_1)\,\widehat{u}(h_2)\,\widehat{u}(h_3)\,\mathrm{d}\sigma(h)\\ &-\int_{S;\,-h_3>h_2>0} i\,h_1\,|h_3|^{2m}\,\big(\Lambda(1,0+)\,-\,\Lambda(-1,0+)\big)\,\widehat{u}(h_1)\,\widehat{u}(h_2)\,\widehat{u}(h_3)\,\mathrm{d}\sigma(h)\,.\end{aligned}$$

Noting that $\lambda := \Lambda(-1,0+)\,-\,\Lambda(1,0+)$ is such that $\overline{\lambda} = \Lambda(1,0-)\,-\,\Lambda(-1,0-)$ by (ii), we see that

$$K_2 \;=\; \frac{1}{2}\int_{S;\,|h_3|>|h_2|} i\,h_1\,|h_3|^{2m}\,\mathrm{sgn}(h_3)\,(\overline{\lambda}-\lambda-(\lambda+\overline{\lambda})\,\mathrm{sgn}(h_2))\,\widehat{u}(h_1)\,\widehat{u}(h_2)\,\widehat{u}(h_3)\,\mathrm{d}\sigma(h)\,.$$

In addition, we observe that for $(h_1,h_2,h_3)\in S$ such that $|h_3|>|h_2|$, $\mathrm{sgn}(h_3)=-\mathrm{sgn}(h_1)$. Therefore

$$K_2 \;=\; \frac{1}{2}\int_{S;\,|h_3|>|h_2|} i\,|h_1|\,|h_3|^{2m}\,(\lambda-\overline{\lambda}+(\lambda+\overline{\lambda})\,\mathrm{sgn}(h_2))\,\widehat{u}(h_1)\,\widehat{u}(h_2)\,\widehat{u}(h_3)\,\mathrm{d}\sigma(h)\,.$$

In particular if $\widehat{u}$ is odd, as in the example used above for te kernel Λ_0,

$$K_2 \;=\; \mathrm{Re}\lambda\int_{S;\,|h_3|>|h_2|} i\,|h_1|\,|h_3|^{2m}\,\mathrm{sgn}(h_2)\,\widehat{u}(h_1)\,\widehat{u}(h_2)\,\widehat{u}(h_3)\,\mathrm{d}\sigma(h)\,,$$

while if $\widehat{u}$ is even,

$$K_2 \;=\; -\,\mathrm{Im}\lambda\int_{S;\,|h_3|>|h_2|} |h_1|\,|h_3|^{2m}\,\widehat{u}(h_1)\,\widehat{u}(h_2)\,\widehat{u}(h_3)\,\mathrm{d}\sigma(h)\,.$$

If we take more specifically $\widehat{u}$ of the form $\widehat{u}(\xi) = i\,\mathrm{sgn}(\xi)U(\xi)$, using again that $\mathrm{sgn}(h_3) = -\mathrm{sgn}(h_1)$ in the region of interest, we find that

$$K_2 \;=\; -\,\mathrm{Re}\lambda \int_{S;\,|h_3|>|h_2|} |h_1|\,|h_3|^{2m}\, U(h_1)\, U(h_2)\, U(h_3)\, \mathrm{d}\sigma(h)\,.$$

By (nv) we know that either $\mathrm{Re}\lambda$ or $\mathrm{Im}\lambda$ is nonzero. If $\mathrm{Re}\lambda \neq 0$, we can take $U = -\left(\mathrm{Re}\lambda\right)^{-1/3} U_n$ with U_n defined - similarly as for the kernel Λ_0 - by

$$U_n(\xi) \;=\; \begin{cases} 1\,, & \text{if } |\xi| \in [0, 1/n], \\ n^{-2m+1/2}\,, & \text{if } |\xi| \in [n^2, n^2+1/n]\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

so that

$$K_2 \;\geq\; \int_{n^2}^{n^2+1/n} \int_{n^2}^{n^2+1/n} n^{2(2m+1)}\, n^{-4m+1} \;=\; n\,.$$

If $\mathrm{Im}\lambda \neq 0$, we can take $u = u_n$ defined instead by $\widehat{u} \;=\; -\left(\mathrm{Im}\lambda\right)^{-1/3} U_n$, and we obtain again that

$$K_2 \;\geq\; \int_{n^2}^{n^2+1/n} \int_{n^2}^{n^2+1/n} n^{2(2m+1)}\, n^{-4m+1} \;=\; n\,.$$

Remembering that $|J_1 + J_2 + K_1| \leq C\,\|u\|_{H^m}^3$, we find that

$$I \;=\; -\,2\pi\,\langle \partial_x^m u_n, \partial_x^{m+1} \mathscr{Q}(u_n)\rangle_{L^2} \;\geq\; n \,-\, C\,\|u_n\|_{H^m}^3 \;\geq\; n/2$$

for n large enough, since $\|u_n\|_{H^m}$ is bounded uniformly in n (in both cases). □

It is not yet clear how the blow-up of inner products should imply ill-posedness, or more precisely the existence of a sequence of (analytic) initial data (u_0^n) going to zero in H^m for which the solutions u_n are such that $\|u_n(t_n)\|_{H^m}$ goes to infinity with t_n going to zero. Regarding this issue for dispersive PDEs (replacing the analytic setting for well-posedness by a subcritical Sobolev one), a very nice method is due to Christ, Colliander and Tao [5, 6, 19]. However, the fact that our inner products behave only as n for functions involving frequencies of order n^2 seems to be a major obstacle to adapt their method to our framework. (For reasons that would be too long to explain here, $O(n^2)$ inner products for $O(n)$ frequencies would be much more favorable, but we do not have such an example.)

4 Ill-posedness in Sobolev spaces: the model case

In this paragraph, we consider the Cauchy problem (1) when the kernel Λ is

$$\Lambda_0(k, \ell) = \mathrm{sgn}(k)\,\mathrm{sgn}(\ell)\,.$$

As already pointed out, for such a kernel we have (i), (ii), (iii), (iv), (nv), and the corresponding nonlocal Burgers equation in (1) reads (for smooth enough functions)

$$\partial_t u - 4\,\pi\,\mathscr{H}(u)\,\partial_x \mathscr{H}(u) \;=\; 0\,.$$

Noting that $\partial_x \mathscr{H} = |\partial_x|$ the Fourier multiplier with symbol $|k|$, we can rewrite (1) for that special kernel as

$$\partial_t u - 4\,\pi\,\mathscr{H}(u)\,|\partial_x| u = 0\,, \quad u_{|t=0} = u_0\,. \tag{9}$$

We recall that the Hilbert transform $\mathscr{H}$ is a continuous operator on every Sobolev space $H^m(\mathbb{R})$. Proposition 1 shows that (9) is locally well-posed for analytic initial data, a dense subset in $H^m(\mathbb{R})$. The following result shows that local well-posedness of (9) in any Sobolev space $H^m(\mathbb{R})$, $m \geq 2$, is linked to some regularity properties of the initial condition. An immediate consequence is that for 'most' initial conditions (9) has no local-in-time solution in $H^m(\mathbb{R})$.

Proposition 3. *Let $u_0 \in H^m(\mathbb{R})$ with $m \in \mathbb{N}$, $m \geq 2$, and let us assume that there exists $T > 0$ and $u \in \mathscr{C}([-T,T]; H^m(\mathbb{R}))$ solution to (9). If moreover $\mathscr{H}(u_0)(0) \neq 0$, then there exists a function $\psi \in \mathscr{C}_0^\infty(\mathbb{R})$ satisfying $\psi(0) = 1$ and $\psi\, u_0 \in H^{m+1/4}(\mathbb{R})$, that is, u_0 has $H^{m+1/4}$ regularity near 0.*

Corollary 1. *For all integer $m \geq 2$, there exists a dense subset $\mathscr{O} \subset H^m(\mathbb{R})$ such that for all $u_0 \in \mathscr{O}$, the Cauchy problem (9) has no solution $u \in \mathscr{C}([-T,T]; H^m(\mathbb{R}))$ for any $T > 0$.*

Proof of Proposition 3. The key is the ellipticity in (t,x) of the equation in (9) when $\mathscr{H}(u)$ does not vanish, which leads to a second order elliptic PDE. Then we can invoke classical elliptic regularity results to show that the local existence of a smooth solution necessarily yields higher smoothness of the 'initial' data. This was already the guideline of [12], and we follow here the main lines of [12, section 3]. We also refer to [14] for similar results.

Let us therefore assume that $u_0 \in H^m(\mathbb{R})$, $m \geq 2$, satisfies $\mathscr{H}(u_0)(0) \neq 0$, and that $u \in \mathscr{C}([-T,T]; H^m(\mathbb{R}))$ is a solution to (9). Applying Lemma 5, which holds in a general context, we get

$$u \in \cap_{j=0}^m \mathscr{C}^j([-T,T]; H^{m-j}(\mathbb{R}))\,,$$

and introducing $v := \partial_x^{m-1} u$, we also have

$$\partial_t v - 4\,\pi\,\mathscr{H}(u)\,|\partial_x| v \in \mathscr{C}([-T,T]; H^1(\mathbb{R})) \cap \mathscr{C}^1([-T,T]; L^2(\mathbb{R}))\,.$$

Applying the operator $\partial_t + 4\,\pi\,\mathscr{H}(u)\,|\partial_x|$, we obtain that[1]

$$\begin{aligned}
&\partial_t^2 v + 16\,\pi^2\,\mathscr{H}(u)^2\,\partial_x^2 v - f_1 - f_2 \in \mathscr{C}([-T,T]; L^2(\mathbb{R}))\,,\\
&f_1 := 4\,\pi\,\mathscr{H}(\partial_t u)\,|\partial_x|\,v\,, \quad f_2 := 16\,\pi^2\,\mathscr{H}(u)\,[\,|\partial_x|\,;\,\mathscr{H}(u)\,]\,|\partial_x|\,v\,.
\end{aligned}$$

Some regularity can be obtained for f_1 and f_2 by applying standard results on Sobolev spaces and commutators. Let us start with f_1. According to the regularity of u, we have $\mathscr{H}(\partial_t u) \in \mathscr{C}([-T,T]; H^1(\mathbb{R}))$ and $|\partial_x|\,v \in \mathscr{C}([-T,T]; L^2(\mathbb{R}))$. We thus obtain $f_1 \in \mathscr{C}([-T,T]; L^2(\mathbb{R}))$ because the product of functions defines a continuous mapping from

[1]The reduction to a second order differential equation is merely based, as in [12], on the 'trick' $|\partial_x|^2 = -\partial_x^2$.

$H^1(\mathbb{R}) \times L^2(\mathbb{R})$ to $L^2(\mathbb{R})$. The regularity of f_2 relies on the following classical inequality for commutators, see e.g. [11, 18]:

$$\|[\,|\partial_x|\,;\, w_1\,]\, w_2\|_{L^2(\mathbb{R})} \lesssim \|w_1\|_{H^2(\mathbb{R})}\, \|w_2\|_{L^2(\mathbb{R})}\,.$$

The commutator $[\,|\partial_x|\,;\, \mathscr{H}(u)\,]\,|\partial_x|\, v$ therefore belongs to $\mathscr{C}([-T,T];L^2(\mathbb{R}))$, and using the same argument as for f_1, we find that $f_2 \in \mathscr{C}([-T,T];L^2(\mathbb{R}))$. Summing up, we have shown

$$\partial_t^2 v + 16\,\pi^2\, \mathscr{H}(u)^2\, \partial_x^2 v = f \in \mathscr{C}([-T,T];L^2(\mathbb{R})) \subset L^2(]-T,T[\,\times\mathbb{R})\,. \tag{10}$$

The function $v \in H^1(]-T,T[\times\mathbb{R})$ can be regarded as a solution to the linear equation (10) which is strongly elliptic in the neighborhood of the point $(t,x) = (0,0)$, and whose source term belongs to $L^2(]-T,T[\times\mathbb{R})$. By standard elliptic regularity theory [7, p. 309], we obtain that there exists a function $\Psi \in \mathscr{C}_0^\infty(\mathbb{R}^2)$ such that $\Psi(0,0) = 1$ and $\Psi\, v \in H^2(\mathbb{R}^2)$. Using the imbeddings $H^2(\mathbb{R}^2) \subset H^{3/4}(\mathbb{R}; H^{5/4}(\mathbb{R})) \subset \mathscr{C}(\mathbb{R}; H^{5/4}(\mathbb{R}))$, we find that $\Psi(0,\cdot)\, \partial_x^{m-1} u_0$ belongs to $H^{5/4}(\mathbb{R})$, from which Proposition 3 follows. □

Proof of Corollary 1. It is based on the following two observations:

- The set of functions $u_0 \in H^m(\mathbb{R})$ such that $\mathscr{H}(u_0)(0) \neq 0$ is an open dense subset of $H^m(\mathbb{R})$, because its complementary set is a closed hyperplane of $H^m(\mathbb{R})$.

- The set of functions $u_0 \in H^m(\mathbb{R})$ such that there exists a function $\psi \in \mathscr{C}_0^\infty(\mathbb{R})$ satisfying $\psi(0) = 1$ and $\psi\, u_0 \in H^{m+1/4}(\mathbb{R})$ is a strict subspace of $H^m(\mathbb{R})$, and thus has empty interior. In other words, its complementary set is dense in $H^m(\mathbb{R})$.

In particular, the set of initial data $u_0 \in H^m(\mathbb{R})$ such that $\mathscr{H}(u_0)(0) \neq 0$ and such that there does not exist a function $\psi \in \mathscr{C}_0^\infty(\mathbb{R})$ satisfying $\psi(0) = 1$ and $\psi\, u_0 \in H^{m+1/4}(\mathbb{R})$ is the intersection of an open dense subset with a dense subset, hence the conclusion. □

5 Ill-posedness in Sobolev spaces: the general case

Let us go back to an abstract kernel Λ satisfying (i)-(ii)-(iii)-(iv)-(nv). For later use we introduce the notations

$$\lambda := \Lambda(1,0+) - \Lambda(-1,0+)\,, \quad \mu := \Lambda(1,0+) + \Lambda(-1,0+)\,.$$

Note that the assumption in (nv) means that λ is a nonzero complex number. Furthermore, it will turn out that $\lambda = \mu$, or equivalently $\Lambda(-1,0+) = 0$, is a remarkable case in that, for kernels having this property, the *principal part* of the nonlocal equation in (1) reduces to the (generalized) complex Burgers equation

$$\frac{1}{2\,\pi}\,\partial_t z + \lambda\, z\, \partial_x z = 0\,, \tag{11}$$

for $z := u + ih$, $h := \mathscr{H}(u)$. In general, in order to identify the principal part of the equation in (1), we first rewrite the operator $\mathscr{Q}$ (formally) by means of the Fourier inverse formula as

$$\mathscr{Q}(u)(x) = \frac{1}{2\,\pi} \iint_{\mathbb{R}^2} \Lambda(\xi,\ell)\, \mathrm{e}^{i\,x\,(\xi+\ell)}\, \widehat{u}(\xi)\, \widehat{u}(\ell)\, \mathrm{d}\xi\, \mathrm{d}\ell\,,$$

so that

$$\partial_x \mathscr{Q}(u)(x) = \int_{\mathbb{R}} m_u(x,\xi)\,\mathrm{e}^{i\,x\,\xi}\,\widehat{u}(\xi)\,\mathrm{d}\xi\,, \quad m_u(x,\xi) \;:=\; \frac{i\,\xi}{\pi}\int_{\mathbb{R}} \Lambda(\xi,\ell)\,\mathrm{e}^{i\,x\,\ell}\,\widehat{u}(\ell)\,\mathrm{d}\ell\,. \tag{12}$$

We need the asymptotics of the symbol m_u for large ξ, which is the purpose of the following results.

Lemma 2. *Let $\Lambda \in L^\infty(\mathbb{R}\times\mathbb{R};\mathbb{C})$ satisfy* (i)-(ii)-(iii)-(iv)*, and u be in $H^s(\mathbb{R})$ with $s > 1/2$. Then*

$$M_u(x,\xi) \;:=\; \frac{1}{\pi}\int \Lambda(\xi,\ell)\,\mathrm{e}^{i\,x\,\ell}\,\widehat{u}(\ell)\,\mathrm{d}\ell$$

is well defined for all $(x,\xi)\in\mathbb{R}\times\mathbb{R}$ and there exists $C_s > 0$ depending only on s such that for all $(x,\xi)\in\mathbb{R}\times\mathbb{R}$

$$|M_u(x,\xi)| \;\leq\; C_s\,\|\Lambda\|_{L^\infty}\,\|u\|_{H^s}\,.$$

Furthermore, there exists $C(s,\Lambda)$ such that

$$|M_u(x,\xi) - M_u^0(x,\xi)| \;\leq\; C(s,\Lambda)\,\|u\|_{H^s}\,\langle\xi\rangle^{-(2s-1)/(2s+1)}\,,$$

where

$$\begin{aligned} M_u^0(x,\xi) := {}& (\Lambda(\mathrm{sgn}(\xi),0+) + \Lambda(\mathrm{sgn}(\xi),0-))\,u(x) \\ & + i\,(\Lambda(\mathrm{sgn}(\xi),0+) - \Lambda(\mathrm{sgn}(\xi),0-))\,\mathscr{H}(u)(x)\,. \end{aligned}$$

Proof. For simplicity we omit the factor $1/\pi$ in the definition of M_u. This symbol is clearly bounded by $C_s\,\|\Lambda\|_{L^\infty}\,\|u\|_{H^s}$ with $C_s = (\int\langle\ell\rangle^{-2s}\mathrm{d}\ell)^{1/2}$. Furthermore, it can be split as

$$M_u(x,\xi) \;=\; \int_{-f(\xi)}^{f(\xi)} \Lambda(\xi,\ell)\,\mathrm{e}^{i\,x\,\ell}\,\widehat{u}(\ell)\,\mathrm{d}\ell \;+\; \int_{|\ell|>f(\xi)} \Lambda(\xi,\ell)\,\mathrm{e}^{i\,x\,\ell}\,\widehat{u}(\ell)\,\mathrm{d}\ell\,,$$

where $f(\xi) > 0$ will be specified later on but will at least be such that $|\xi| \gg f(\xi) \gg 1$ when $|\xi|$ tends to infinity. By the Cauchy–Schwarz inequality, we have

$$\begin{aligned} \left|\int_{|\ell|>f(\xi)} \Lambda(\xi,\ell)\,\mathrm{e}^{i\,x\,\ell}\,\widehat{u}(\ell)\,\mathrm{d}\ell\right| &\leq \|\Lambda\|_{L^\infty}\,\|u\|_{H^s}\,\left(\int_{|\ell|>f(\xi)}\langle\ell\rangle^{-2s}\,\mathrm{d}\ell\right)^{1/2} \\ &\lesssim \|\Lambda\|_{L^\infty}\,\|u\|_{H^s}\,\langle f(\xi)\rangle^{-s+1/2}\,. \end{aligned}$$

As to the first integral, we can split it again as

$$\int_{-f(\xi)}^{0} \Lambda(\xi,\ell)\,\mathrm{e}^{i\,x\,\ell}\,\widehat{u}(\ell)\,\mathrm{d}\ell \;+\; \int_{0}^{f(\xi)} \Lambda(\xi,\ell)\,\mathrm{e}^{i\,x\,\ell}\,\widehat{u}(\ell)\,\mathrm{d}\ell\,,$$

and deal with each term separately. The latter can be written as

$$\begin{aligned} \int_0^{f(\xi)} \Lambda(\xi,\ell)\,\mathrm{e}^{i\,x\,\ell}\,\widehat{u}(\ell)\,\mathrm{d}\ell = {}& \Lambda(\mathrm{sgn}(\xi),0+)\int_0^{+\infty}\mathrm{e}^{i\,x\,\ell}\,\widehat{u}(\ell)\,\mathrm{d}\ell \\ & + \int_0^{f(\xi)} (\Lambda(\mathrm{sgn}(\xi),\ell/|\xi|) \;-\; \Lambda(\mathrm{sgn}(\xi),0+))\,\mathrm{e}^{i\,x\,\ell}\,\widehat{u}(\ell)\,\mathrm{d}\ell \\ & - \Lambda(\mathrm{sgn}(\xi),0+)\int_{f(\xi)}^{+\infty}\mathrm{e}^{i\,x\,\ell}\,\widehat{u}(\ell)\,\mathrm{d}\ell\,, \end{aligned}$$

where the first term is the one contributing to M_u^0, the last one is bounded again by a constant times $\|\Lambda\|_{L^\infty}\,\|u\|_{H^s}\,\langle f(\xi)\rangle^{-s+1/2}$, and the middle one is bounded by $C(\Lambda)\,\|\widehat{u}\|_{L^1}\,f(\xi)/|\xi|$, with $C(\Lambda)$ a Lipschitz constant for Λ on the line segment joining $(\operatorname{sgn}(\xi),0)$ to $(\operatorname{sgn}(\xi),1)$. Using that $\|\widehat{u}\|_{L^1}\lesssim\|u\|_{H^s}$ for $s>1/2$, we thus find the same estimate as for the other remainder terms provided that

$$f(\xi)\lesssim|\xi|\,\langle f(\xi)\rangle^{-s+1/2}\,,$$

which is the case for $f(\xi)=|\xi|^{1/(s+1/2)}$. The other integral

$$\begin{aligned}\int_{-f(\xi)}^{0}\Lambda(\xi,\ell)\,\mathrm{e}^{i\,x\,\ell}\,\widehat{u}(\ell)\,\mathrm{d}\ell=\Lambda(\operatorname{sgn}(\xi),0-)&\int_{-\infty}^{0}\mathrm{e}^{i\,x\,\ell}\,\widehat{u}(\ell)\,\mathrm{d}\ell\\&+\int_{-f(\xi)}^{0}(\Lambda(\operatorname{sgn}(\xi),\ell/|\xi|)\;-\;\Lambda(\operatorname{sgn}(\xi),0-))\,\mathrm{e}^{i\,x\,\ell}\,\widehat{u}(\ell)\,\mathrm{d}\ell\\&-\Lambda(\operatorname{sgn}(\xi),0-)\int_{-\infty}^{-f(\xi)}\mathrm{e}^{i\,x\,\ell}\,\widehat{u}(\ell)\,\mathrm{d}\ell\,,\end{aligned}$$

is dealt with in the same manner. Eventually, we find that the 'principal part' of M_u is

$$M_u^0(x,\xi)\;=\;\Lambda(\operatorname{sgn}(\xi),0+)\int_0^{+\infty}\mathrm{e}^{i\,x\,\ell}\,\widehat{u}(\ell)\,\mathrm{d}\ell\;+\;\Lambda(\operatorname{sgn}(\xi),0-)\int_{-\infty}^{0}\mathrm{e}^{i\,x\,\ell}\,\widehat{u}(\ell)\,\mathrm{d}\ell\,,$$

which can be rewritten as claimed by a straightforward manipulation using the inverse Fourier formulas

$$u(x)=\frac{1}{2\,\pi}\int\mathrm{e}^{i\,x\,\ell}\,\widehat{u}(\ell)\,\mathrm{d}\ell\,,\quad i\,\mathscr{H}(u)(x)=\frac{1}{2\,\pi}\int\mathrm{e}^{i\,x\,\ell}\operatorname{sgn}(\ell)\,\widehat{u}(\ell)\,\mathrm{d}\ell\,.$$

□

In the estimate of the remainder 'symbol', the exponent $(2s-1)/(2s+1)$ is less than one. However, if we assume more regularity on u, we can achieve a decay of order one.

Lemma 3. *For $s>3/2$, there exists $C=C(s,\Lambda)$ such that for all $u\in H^s(\mathbb{R})$,*

$$|M_u(x,\xi)-M_u^0(x,\xi)|\;\leq\;C\,\|u\|_{H^s}\,\langle\xi\rangle^{-1}\,.$$

Proof. We proceed similarly as in the proof of Lemma 2, but we choose $f(\xi)=|\xi|^{1/(s-1/2)}$, which is indeed $o(|\xi|)$ if $s>3/2$ and such that $\langle f(\xi)\rangle^{-s+1/2}=O(|\xi|^{-1})$ when $|\xi|$ goes to infinity, and we deal with the middle term in the following, slightly different way. Indeed, by the mean value theorem and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\left|\int_0^{f(\xi)}(\Lambda(\operatorname{sgn}(\xi),\ell/|\xi|)\;-\;\Lambda(\operatorname{sgn}(\xi),0+))\,\mathrm{e}^{i\,x\,\ell}\,\widehat{u}(\ell)\,\mathrm{d}\ell\right|\\&\qquad\qquad\leq C(\Lambda)\,|\xi|^{-1}\left(\int\langle\ell\rangle^{-2(s-1)}\,\mathrm{d}\ell\right)^{1/2}\|u\|_{H^s}\,.\end{aligned}$$

□

Now, if we define $m_u^0(x,\xi) := i\,\xi\, M_u^0(x,\xi)$, we clearly have (as for m_u) that m_u^0 is positively homogeneous degree one in ξ, and

$$m_u^0(x,-\xi) = \overline{m_u^0(x,\xi)}\,,$$

so that

$$m_u^0(x,\xi) \;=\; |\xi|\,\mathrm{Re}\, m_u^0(x,1) \;+\; i\,\xi\,\mathrm{Im}\, m_u^0(x,1)\,.$$

This in turn yields the formula

$$m_u^0(x,\xi) \;=\; -\,|\xi|\,\mathrm{Im}(\lambda\, z(x)) \;+\; i\,\xi\,\mathrm{Re}(\mu\, z(x))\,, \tag{13}$$

where again $z = u + ih$, $h = \mathscr{H}(u)$. Therefore, the principal equation

$$\partial_t u \;+\; \partial_x \mathscr{Q}^0(u) \;=\; 0\,,$$

with

$$\partial_x \mathscr{Q}^0(u)(x) := \int m_u^0(x,\xi)\,\mathrm{e}^{i\,x\,\xi}\,\widehat{u}(\xi)\,\mathrm{d}\xi\,,$$

amounts to

$$\frac{1}{2\pi}\,\partial_t u \;-\; \mathrm{Im}\,(\lambda\, z)\,|\partial_x| u \;+\; \mathrm{Re}\,(\mu\, z)\,\partial_x u \;=\; 0\,. \tag{14}$$

Recalling that $|\partial_x| u = \partial_x h$, it is tempting to derive from this equation a system for (u,h) (or equivalently a complex equation for z). Applying the Hilbert transform to (14) and using the identities

$$\mathscr{H}\partial_x = |\partial_x|\,,\quad \mathscr{H}|\partial_x| = -\partial_x\,,\quad \mathscr{H}[uv + \mathscr{H}(u)\mathscr{H}(v)] = u\mathscr{H}(v) + v\mathscr{H}(u)\,,$$

we deduce from (14) that

$$\frac{1}{2\pi}\,\partial_t h \;+\; \mathrm{Im}(\lambda\, z)\,|\partial_x| u \;+\; \mathrm{Re}(\mu\, z)\,\partial_x h \;+\mathrm{Re}((\mu-\lambda)\,\mathscr{H}(z\,\partial_x u)) \;=\; 0\,. \tag{15}$$

This is where the special case $\lambda = \mu$ arises, because if $\lambda = \mu$ then the system (14)-(15) is easily seen to be equivalent to the complex Burgers equation (11). However, in general, that system is not 'closed' (in the sense of physicists).

Before proving Theorem 1, we need a 'quantitative' result on the difference between the bilinear operator $\partial_x \mathscr{Q}$ and its principal part.

Proposition 4. *The bilinear mapping*

$$(u,v) \longmapsto \left(x \mapsto \int_{\mathbb{R}} \mathrm{e}^{i\,x\,\xi}\,(m_u(x,\xi) - m_u^0(x,\xi))\,\widehat{v}(\xi)\,\mathrm{d}\xi \right)\,,$$

is continuous on $H^3(\mathbb{R}) \times L^2(\mathbb{R})$ with values in $L^2(\mathbb{R})$, and is also continuous on $H^4(\mathbb{R}) \times H^1(\mathbb{R})$ with values in $H^1(\mathbb{R})$. (Recall that m_u and m_u^0 are given in (12) *and* (13)*.)*

Proof. Let us define the symbol

$$r(x,\xi) := m_u(x,\xi) - m_u^0(x,\xi)\,.$$

Lemma 3 shows that r is bounded on $\mathbb{R}^2$ by a constant times $\|u\|_{H^2}$. Moreover, the space derivative $\partial_x r$ is obtained by changing u into u' in the definition of r. Therefore the derivative $\partial_x r$ is bounded on $\mathbb{R}^2$ by a constant times $\|u\|_{H^3}$. Let us now look at the derivative $\partial_\xi r$. It is clear that $\partial_\xi m_u^0$ is a bounded function on $\mathbb{R}^2$ whose L^∞ norm is controlled by $\|u\|_{H^1}$. In the same way, the second derivative $\partial_x\partial_\xi m_u^0$ is a bounded function on $\mathbb{R}^2$ whose L^∞ norm is controlled by $\|u\|_{H^2}$. Let us now examine the derivative $\partial_\xi m_u$ (in the sense of distributions). This is where we shall use the continuity assumption in (iv'). Taking a test function φ and integrating by parts on each subset $\{\xi \gtrless 0\}$ in

$$\pi \int_{\mathbb{R}^2} m_u(x,\xi)\,\partial_\xi\varphi(x,\xi)\,\mathrm{d}x\,\mathrm{d}\xi \;=\; \int_{\mathbb{R}^2}\int_{\mathbb{R}} \mathrm{e}^{i\,x\,\ell}\,i\,\xi\,\Lambda(\xi,\ell)\,\widehat{u}(\ell)\,\partial_\xi\varphi(x,\xi)\,\mathrm{d}\ell\,\mathrm{d}x\,\mathrm{d}\xi\,,$$

we find that $\partial_\xi m_u$ coincides with the function

$$(x,\xi) \longmapsto \int_{\mathbb{R}} \mathrm{e}^{i\,x\,\ell}\left(i\,\Lambda(\xi,\ell) + i\,\xi\,\partial_1\Lambda(\xi,\ell)\right)\widehat{u}(\ell)\,\mathrm{d}\ell\,,$$

where $\partial_1\Lambda(\xi,\ell)$ denotes the (classical) derivative of Λ with respect to ξ for $\xi \neq 0$. It is not difficult to check that by the assumptions (iii)-(iv)-(iv'), the function $(\xi,\ell) \mapsto \xi\,\partial_1\Lambda(\xi,\ell)$ is bounded. Therefore, by the Cauchy-Schwarz inequality we find that $\partial_\xi m_u$ is bounded on $\mathbb{R}^2$, and its L^∞ norm is estimated by a constant times $\|u\|_{H^1}$. (With Λ discontinuous across the second diagonal $\{k+\ell=0\}$, the derivative $\partial_\xi m_u$ would involve a term of the form $\widehat{u}(-\xi)$ which would not be necessarily bounded.) In the same way, $\partial_x\partial_\xi m_u$ is a bounded function on $\mathbb{R}^2$ whose L^∞ norm is estimated by a constant times $\|u\|_{H^2}$. Summing up, the functions $r, \partial_x r, \partial_\xi r, \partial_x\partial_\xi r$ are bounded and their L^∞ norms are controlled by $\|u\|_{H^3(\mathbb{R})}$. These are all the ingredients required to show the boundedness on $L^2(\mathbb{R})$ of the pseudodifferential operator with symbol r, see [10]. To prove the continuity property on $H^1(\mathbb{R})$, it is sufficient to differentiate under the integral and to apply the preceeding analysis. □

Let us now turn to the proof of Theorem 1. Assume we have a solution $u \in \mathscr{C}([-T,T];H^m(\mathbb{R}))$, $m \geq 4$ of (1). Lemma 5 shows that $v := \partial_x^{m-1}u \in \mathscr{C}([-T,T];H^1(\mathbb{R})) \cap \mathscr{C}^1([-T,T];L^2(\mathbb{R}))$ satisfies

$$\partial_t v + 2\,\mathscr{B}(u,\partial_x v) \in \mathscr{C}([-T,T];H^1(\mathbb{R})) \cap \mathscr{C}^1([-T,T];L^2(\mathbb{R}))\,.$$

Observing that $2\,\mathscr{B}(u,\partial_x v)$ coincides with

$$\int_{\mathbb{R}} m_u(x,\xi)\,\mathrm{e}^{i\,x\,\xi}\,\widehat{v}(\xi)\,\mathrm{d}\xi\,,$$

we apply Proposition 4 and obtain that

$$\frac{1}{2\,\pi}\,\partial_t v - \mathrm{Im}(\lambda\,z)\,|\partial_x| v + \mathrm{Re}(\mu\,z)\,\partial_x v \in \mathscr{C}([-T,T];H^1(\mathbb{R})) \cap \mathscr{C}^1([-T,T];L^2(\mathbb{R}))\,. \quad (16)$$

By a similar 'trick' as in Section 4, we can make the ellipticity of (16) more evident. Applying the operator

$$\frac{1}{2\pi}\,\partial_t \,+\, \mathrm{Im}(\lambda\, z)\,|\partial_x| \,+\, \mathrm{Re}(\mu\, z)\,\partial_x$$

to (16) we get indeed the second order PDE[2]

$$\frac{1}{4\,\pi^2}\,\partial_t^2 u \,+\, \frac{1}{\pi}\,\mathrm{Re}(\mu\, z)\,\partial_t\partial_x u \,+\, \big((\mathrm{Im}(\lambda\, z))^2 + (\mathrm{Re}(\mu\, z))^2\big)\,\partial_x^2 u \,=\, f\,,$$

where the source term f belongs to $\mathscr{C}([-T,T];L^2(\mathbb{R}))$. The ellipticity of the latter equation at $(t,x)=(0,0)$ is ensured by choosing an initial condition u_0 satisfying

$$\mathrm{Im}\big(\lambda\,(u_0(0) + i\,\mathscr{H}(u_0)(0))\big) \neq 0\,. \tag{17}$$

If it is the case then we can apply the same arguments as in Section 4, which completes the proof of Theorem 1.

Observe that the condition in (17) merely coincides with $\mathscr{H}(u_0)(0)\neq 0$ in the model case (3).

Appendix

Lemma 4. *Consider the polynomials* $P(x_1,x_2,x_3) \,=\, x_1\,x_3^{2m} + x_3\,x_1^{2m}$ *and* $A(x_1,x_2,x_3) \,=\, x_1+x_2+x_3$. *Then*

$$P \,=\, (-1)^m\,\Big(x_1\,x_2^m\,x_3^m \,+\, x_1^m\,x_2^m\,x_3 \,-\, x_1^m\,x_2\,x_3^m \\ +\sum_{k=2}^{m-1}\binom{m}{k}\,x_2^k\,(\,x_1^{m+1-k}\,x_3^m \,+\, x_1^m\,x_3^{m+1-k}\,)\Big) \bmod A\,.$$

Proof. We first note that for any polynomial B and any integer k,

$$(A-B)^k \,=\, (-1)^k\,B^k \bmod A\,.$$

Therefore,

$$\begin{aligned}P &= x_1\,x_3^m\,(A-(x_1+x_2))^m \,+\, x_3\,x_1^m\,(A-(x_3+x_2))^m\\ &= (-1)^m\,\left(x_1\,x_3^m\,(x_1+x_2)^m \,+\, x_3\,x_1^m\,(x_3+x_2)^m\right) \bmod A\\ &= (-1)^m\,\left(\sum_{k=1}^{m}\binom{m}{k}\,x_2^k\,(\,x_1^{m+1-k}\,x_3^m \,+\, x_1^m\,x_3^{m+1-k}\,) \,+\, x_1^{m+1}\,x_3^m \,+\, x_3^{m+1}\,x_1^m\right) \bmod A\\ &= (-1)^m\,\left(\sum_{k=1}^{m}\binom{m}{k}\,x_2^k\,(\,x_1^{m+1-k}\,x_3^m \,+\, x_1^m\,x_3^{m+1-k}\,) \,-\, x_1^m\,x_2\,x_3^m\right) \bmod A\,.\end{aligned}$$

□

[2] The control of commutators is entirely similar to Section 4 so we omit the details.

Lemma 5. *Let $\Lambda \in L^\infty(\mathbb{R}^2;\mathbb{C})$ satisfy conditions* (i), (ii)*, and let the bilinear operator $\mathscr{B}$ be defined by* (6)*. Then for all $m \geq 2$, if $u \in \mathscr{C}([-T,T];H^m(\mathbb{R}))$, there holds*

$$u \in \cap_{j=0}^{m} \mathscr{C}^j([-T,T];H^{m-j}(\mathbb{R}))\,,$$

and

$$\partial_t(\partial_x^{m-1}u) + 2\,\mathscr{B}(u,\partial_x^m u) \in \mathscr{C}([-T,T];H^1(\mathbb{R})) \cap \mathscr{C}^1([-T,T];L^2(\mathbb{R}))\,.$$

Proof. The first part (regularity of u) follows from the continuity properties of $\mathscr{B}$. As shown in [2], if $\Lambda \in L^\infty(\mathbb{R}^2;\mathbb{C})$ satisfies conditions (i), (ii), then $\mathscr{B}$ is a bilinear symmetric continuous operator on $H^n(\mathbb{R}) \times H^n(\mathbb{R})$ with values in $H^n(\mathbb{R})$ for all $n \geq 1$. Furthermore, $\mathscr{B}$ is continuous on $H^1(\mathbb{R}) \times L^2(\mathbb{R})$ with values in $L^2(\mathbb{R})$. Hence $\mathscr{Q} : u \mapsto \mathscr{B}(u,u)$ is a $\mathscr{C}^\infty$ map from $H^n(\mathbb{R})$ to $H^n(\mathbb{R})$ for all $n \geq 1$. The regularity of u follows by a straightforward induction argument.

Let us now compute the equation satisfied by $\partial_x^{m-1}u$. Apllying ∂_x^{m-1} to (1) and using Leibnitz' rule, we get

$$\partial_t(\partial_x^{m-1}u) + 2\,\mathscr{B}(u,\partial_x^m u) = -\sum_{j=1}^{m-1} \tbinom{m}{k} \mathscr{B}(\partial_x^j u, \partial_x^{m-j} u)\,.$$

In the sum on the right hand-side, all terms $\partial_x^j u, \partial_x^{m-j} u$ belong to $\mathscr{C}([-T,T];H^1(\mathbb{R}))$ so the sum belongs to $\mathscr{C}([-T,T];H^1(\mathbb{R}))$. If j does not equal 1 nor $m-1$, then $\partial_x^j u, \partial_x^{m-j} u$ belong to $\mathscr{C}^1([-T,T];H^1(\mathbb{R}))$ and so does $\mathscr{B}(\partial_x^j u, \partial_x^{m-j} u)$. It therefore only remains to prove $\mathscr{B}(\partial_x u, \partial_x^{m-1} u) \in \mathscr{C}^1([-T,T];L^2(\mathbb{R}))$. Since $\partial_x u \in \mathscr{C}^1([-T,T];H^1(\mathbb{R}))$ (use $m \geq 2$), and $\partial_x^{m-1} u \in \mathscr{C}([-T,T];H^1(\mathbb{R})) \cap \mathscr{C}^1([-T,T];L^2(\mathbb{R}))$, it is a simple calculus exercise to show $\mathscr{B}(\partial_x u, \partial_x^{m-1} u) \in \mathscr{C}^1([-T,T];L^2(\mathbb{R})$ (use the continuity properties of $\mathscr{B}$ recalled above). $\square$

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